

EQUATIONS OF TOURNAMENTS ARE NOT FINITELY BASED

J. JEŽEK, P. MARKOVIĆ, M. MARÓTI AND R. MCKENZIE

ABSTRACT. The aim of this paper is to prove that there is no finite basis for the equations satisfied by tournaments. This solves a problem posted in Müller, Nešetřil and Pelant [9].

0. INTRODUCTION

By a tournament we mean a directed graph with loops, such that for any two distinct vertices a and b exactly one of the two cases, either $a \rightarrow b$ or $b \rightarrow a$, takes place.

For any tournament T we can define multiplication on T by setting $ab = ba = b$ whenever $a \rightarrow b$. With respect to this multiplication, T becomes a groupoid (a universal algebra with one binary operation). Moreover, T is uniquely determined by this multiplication. As it is easy to see, the class of groupoids obtained from tournaments in this natural way is just the class of commutative groupoids satisfying $ab \in \{a, b\}$ for all a and b . (One could equivalently say: commutative groupoids, every subset of which is a subgroupoid.) Because of the one-to-one correspondence, we will identify tournaments with their corresponding groupoids. So, a tournament is a commutative groupoid satisfying $ab \in \{a, b\}$ for all a and b . For a tournament T , we have $a \rightarrow b$ if and only if $ab = b$.

A complete bibliography on algebraic representations of tournaments would include the papers [2], [3], [4], [5], [8] and [9].

One can easily check that tournaments satisfy, for example, the following equations:

- (1) $xx = x$
- (2) $xy = yx$
- (3) $x((xy)(xz)) = (xy)(xz)$
- (4) $((xy)z)y = ((xz)y)z$
- (5) $((xy)(xz))((xy)(yz)) = (xy)z$

(On the other hand, the associative law is not satisfied.) It is natural to ask whether a list of equations like this one is complete in the sense that any equation, satisfied in all tournaments, would be derivable. Our main result, Theorem 3, states that not only the five-item list is not complete, but there is no finite complete list of equations for tournaments at all. That question has been first formulated in Müller, Nešetřil and Pelant [9].

1991 *Mathematics Subject Classification.* 05C20, 08B05.

While working on this paper the authors were partially supported by the NSF grant No. DMS-9596043.

1. UNIVERSAL ALGEBRAIC BACKGROUND

For the basics of universal algebra, the reader is referred to either [7] or [1]. We are going to recall here only a few facts that are essential for the proof of our main result.

A variety is a class of (general) algebras of the same similarity type that can be defined by a set of equations. A variety is called finitely based if there is a finite set B of equations satisfied in V , such that every equation satisfied in V is a (logical) consequence of B . An arbitrary class of algebras (such as the class of tournaments) is called finitely based if it generates a finitely based variety.

A variety is said to be locally finite if any finitely generated algebra in V is finite. A variety is locally finite if and only if its free algebras on n generators, for any positive integer n , are all finite.

For a variety V and a positive integer n , we denote by V^n the variety of algebras determined by the equations in at most n variables that are satisfied in V . In this way we obtain a chain of varieties for any given variety V :

$$V^1 \supseteq V^2 \supseteq V^3 \supseteq \dots \supseteq V.$$

It is not difficult to see that an algebra belongs to V^n if and only if all its subalgebras generated by at most n elements belong to V .

One can easily prove that a locally finite variety V of algebras of a finite similarity type is finitely based if and only if $V = V^n$ for at least one positive integer n .

In order to be able to apply this characterization to the variety of groupoids generated by tournaments, we need to know that the variety is locally finite. This will follow from the following observation.

Lemma 1. *Let K be a class of finite algebras of a finite similarity type, closed under forming of subalgebras. The variety V generated by K is locally finite if and only if for every positive integer n there are, up to isomorphism, only finitely many n -generated algebras in K .*

Proof. If V is locally finite, then (for any n) the free n -generated algebra in V is finite, so it has (up to isomorphism) only finitely many homomorphic images; these include all the n -generated algebras in V .

In order to prove the converse, denote by F_n the algebra of terms over a set of n variables, and by E the set of all the ordered pairs (t, s) of elements of F_n that represent an equation $t = s$ satisfied in V . Then E is a congruence of F_n and F_n/E is the free n -generated algebra in V . An equation in n variables belongs to E if and only if it is satisfied in all algebras in K , but we only need to check the at most n -generated ones. If the set S of the n -generated algebras in K is finite (it is sufficient to consider just the nonisomorphic ones), then E has only finitely many blocks, since every block is uniquely determined by a function, assigning to any algebra $A \in S$ and any interpretation of the n variables in A an element of A ; consequently, the free algebra F_n/E is finite. \square

The variety generated by tournaments will be denoted by \mathbf{T} .

Corollary 2 (Crvenković and Marković [2]). *The variety \mathbf{T} is locally finite.* \square

Because of the lack of associativity, we need to distinguish between expressions like $(xy)z$ and $x(yz)$. In order to avoid using too many parentheses, let us make

the following convention: if parentheses are missing, they are always assumed to be grouped to the left. So, for example, $xy(z(uv)t)$ stands for $(xy)((z(uv))t)$.

2. THE VARIETY \mathbf{T} IS NOT FINITELY BASED

Theorem 3. *For every $n \geq 3$ there exists a groupoid M_n with $n+2$ elements such that M_n belongs to \mathbf{T}^n but not to \mathbf{T}^{n+1} . Consequently, the variety \mathbf{T} (the variety generated by tournaments) is not finitely based.*

Proof. Put $M_n = \{a, b_0, \dots, b_n\}$ and define commutative and idempotent multiplication on M_n by

- (1) $ab_1 = b_0$,
- (2) $ab_i = b_i$ for $i \leq n-1$ and $i \neq 1$,
- (3) $ab_n = a$,
- (4) $b_i b_{i+1} = b_i$ for $i < n-1$,
- (5) $b_n b_{n-1} = b_n$,
- (6) $b_i b_j = b_{\max(i,j)}$ for $|i-j| \geq 2$ and $i, j < n$,
- (7) $b_n b_i = b_i$ for $i < n-1$;

the other cases are given by commutativity and idempotency (see also Fig. 1).

Define terms $t_1, s_1, t_2, s_2, \dots, t_n, s_n$ in $n+1$ variables x, y_1, \dots, y_n as follows:

- (1) $t_1 = y_1$ and $s_1 = xy_1$;
- (2) $t_i = s_{i-1}y_i$ and $s_i = t_{i-1}y_i$ for $2 \leq i \leq n-1$;
- (3) $t_n = t_{n-1}y_{n-3}y_n t_{n-1}$ and $s_n = s_{n-1}y_{n-3}y_n t_{n-1}$ if $n \geq 4$, while $t_3 = t_2 s_1 y_3 t_2$ and $s_3 = s_2 s_1 y_3 t_2$ if $n = 3$.

Finally, put $t = s_1 t_n s_n t_n (x t_n)$ and $s = t(s_1 t_n)$.

We are going to prove that the equation $t = s$ is satisfied in any tournament. There will be no confusion if we do not distinguish between a term and its value in a tournament under an interpretation. We distinguish two cases:

If $s_1 = x$, then

$$t = x t_n s_n t_n (x t_n)$$

and

$$s = x t_n s_n t_n (x t_n)(x t_n) = x t_n s_n t_n (x t_n) = t.$$

The other case is $s_1 = y_1$. Then we have $t_1 = s_1, t_2 = s_2, \dots, t_n = s_n$. Consequently,

$$(*) \quad t = y_1 t_n (x t_n) \quad \text{and} \quad s = y_1 t_n (x t_n)(y_1 t_n);$$

clearly, these two values are equal.

So, $t = s$ in every tournament under any interpretation.

This means that the equation $t = s$ is satisfied in \mathbf{T} . On the other hand, we are going to show that the equation is not satisfied in the groupoid M_n . Consider the interpretation $x \rightarrow a, y_i \rightarrow b_i$. By induction on $i = 1, \dots, n$ we can see that $t_i \rightarrow b_i$ and $s_i \rightarrow b_{i-1}$. So, $t \rightarrow a$ and $s \rightarrow b_0$. Since $a \neq b_0$, the equation $t = s$ is not satisfied in M_n .

We have proved that the groupoid M_n does not belong to \mathbf{T} . Since it is generated by $n+1$ elements, it follows that it does not belong to \mathbf{T}^{n+1} . In order to prove that it belongs to \mathbf{T}^n , it is sufficient to show that every subgroupoid of M_n generated by at most n elements belongs to \mathbf{T} .

If we remove either a or b_1 from M_n , we obtain a subtournament. If we remove b_0 , we must remove either a or b_1 in order to obtain a subgroupoid. So, it is sufficient to prove that, for any $i = 2, \dots, n$, $M_n - \{b_i\}$ is a subgroupoid belonging to \mathbf{T} . One can easily check that there are two congruences C_1 and C_2 of $M_n - \{b_i\}$ with trivial intersection, such that both factors $(M_n - \{b_i\})/C_1$ and $(M_n - \{b_i\})/C_2$ are tournaments: C_1 is the congruence generated by (a, b_0) and C_2 is the congruence generated by (b_1, b_0) . (It is easy to see that $\{a, b_0\}$ and $\{b_{i-1}, \dots, b_0\}$ are the only non-singleton blocks of C_1 and C_2 , respectively.) Consequently, $M_n - \{b_i\}$ is a subdirect product of two tournaments (its factor groupoids by C_1 and C_2) and hence belongs to \mathbf{T} . \square

3. GENERALIZATION TO DIRECTED GRAPHS

The one-to-one correspondence between tournaments and commutative groupoids satisfying $ab \in \{a, b\}$ can be naturally extended to a one-to-one correspondence between arbitrary directed graphs with loops (i.e., reflexive binary relations) and arbitrary groupoids satisfying $ab \in \{a, b\}$ for all a and b : we set $ab = b$ if $a \rightarrow b$, and $ab = a$ in the other case. These groupoids are called quasitrivial in some papers, e.g., in [6]. Here we will call them digraphs and identify them with directed graphs with loops. Digraphs are precisely the groupoids such that every subset is a subgroupoid. For two elements of a digraph, $a \rightarrow b$ if and only if $ab = b$.

The variety generated by digraphs will be denoted by \mathbf{D} . This variety is again locally finite, according to Lemma 1. The proof of Theorem 3 works, with the same algebras M_n , even for this non-commutative case. There are only two changes to be made:

First of all, replace the definition of s_1 with $s_1 = xy_1x$. The second change is that the equality of t and s in (*) is not as clear as in the commutative case. But it is again true. It follows from the fact that if $s_1 = y_1$, then $xy_1 = y_1x = y_1$ and the implication

$$xy = yx = y \implies (yz)(xz)(yz) = (yz)(xz)$$

is true in any digraph, which can be easily verified.

So, we know that also digraphs have no finite basis for their equations. In fact, we have proved more:

Theorem 4. *Let V be any variety contained in \mathbf{D} and containing \mathbf{T} . Then V is not finitely based.* \square

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VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37240